INVESTIGATION 1

Exponential Growth Patterns

INVESTIGATION 1 PLANNING CHART

Materials for All Investigations: calculators; student notebooks; colored pens, pencils, or markers; large chart paper

Implementation	Key Terms	Materials	Resources
Problem 1.1 Groups of 2 students **Pacing** 1 day		*For each student* • **Learning Aid 1.1:** Number of Ballots	
Problem 1.2 Groups of 2 students **Pacing** 2 days	standard form expanded form growth factor exponential functions	*For each pair of students* • **Learning Aid 1.2A:** Montarek Chessboard *For each student* • **Learning Aid 1.2B:** Different Reward Plans	
Problem 1.3 Groups of 2 students **Pacing** 1 day	scientific notation	*For each student* • **Learning Aid 1.3:** Garter Snake Population Growth	*For the class* • **Teaching Aid 1.3:** Scientific Notation
Problem 1.4 Think, Pair, Share **Pacing** 1 day	initial value	*For the class* • large poster paper (*optional*)*	
Mathematical Reflection Whole Class Individual Notes **Pacing** $\frac{1}{2}$ day			
Assessment Checkup 1 Individual **Pacing** $\frac{1}{2}$ day		*For each student* • Checkup 1	

*not included in Classroom Materials Kit

PROBLEM 1.1

Making Ballots: Linear or Nonlinear Relationship?

At a Glance

This problem introduces a nonlinear function called an exponential function. In the Initial Challenge, students investigate the growth in the number of ballots created by repeatedly cutting a piece of paper in half. By examining the table, they see that, as the number of cuts increases by 1, the number of ballots doubles, or increases by a factor of 2. In the What If . . . ? situation, they create a graph and write an equation to represent the doubling pattern.

Arc of Learning™
Introduction
Exploration

NOW WHAT DO YOU KNOW?

What are the variables in this situation? How are they related? How is this relationship shown in a table, graph, and equation?

Key Terms	Materials	
	For each student • **Learning Aid 1.1:** Number of Ballots	**Pacing** 1 day **Groups** 2 students **A** 1–2 **C** 11–14 **E** 26–27

Note: If you have a Grade 8 Classroom Materials Kit, please refer to A *Guide to Connected Mathematics®* 4 for a detailed list of materials included or items you will need to prepare ahead of time.

For more on the Teacher Moves listed here, refer to the General Pedagogical Strategies and the Attending to Individual Learning Needs Framework in *A Guide to Connected Mathematics®* 4.

	Facilitating Discourse	Teacher Moves
LAUNCH	CONNECTING TO PRIOR KNOWLEDGE Remind students that in *Thinking with Mathematical Models*, they looked at two kinds of functions, linear functions and inverse variation. **Suggested Questions** • Give an example of each function. • What are the variables? • What is the pattern of change for each function? • Describe the graphs. PRESENTING THE CHALLENGE Describe the situation with Chen creating ballots for the Student Government Association election. **Suggested Questions** • How can you predict the number of ballots after 8 cuts? • How can you predict how many cuts to make if we need 128 ballots? Later they can compare their predictions to their results from the work in the problem.	Make Predictions

<table>
<tr><th></th><th>Facilitating Discourse</th><th>Teacher Moves</th></tr>
<tr><td>EXPLORE</td><td>
PROVIDING FOR INDIVIDUAL NEEDS
Have paper and scissors available for students to cut the ballots, if needed to see and understand the relationship between the number of cuts and the number of ballots created.
Encourage students to look for the multiplicative pattern in the table. When students are working on the What If . . . ?, encourage them to look at Monroe's or Dawson's claim.
Suggested Questions
• How did you find each of the entries in your table?
• What is the relationship between this number of ballots and the previous number of ballots?
• Explain that relationship in terms of the number of cuts.
• How do you create a graph for this data?
• How could describing the pattern help you to write the equation?
PLANNING FOR THE SUMMARY
As students work, look for interesting strategies to share during the summary. Students might be using exponential notation rather than listing all the factors of 2 that are needed for each cut. Some will begin to reason using the general relationship between cuts and number of ballots.
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Assign partner groups to a claim in What If . . . ? Situation A. Differentiate based on the needs of the partner groups.
Agency, Identity, Ownership
</td></tr>
<tr><td>SUMMARIZE</td><td>
DISCUSSING SOLUTIONS AND STRATEGIES
Have students share their strategies and answers for the problem. Lead a discussion around exponents. Display a table from one student group.
Suggested Questions
• How did you get the number of ballots for 5 cuts?
Add a third column to the table, and illustrate each calculation, showing each factor of 2. Stop after showing the calculation for 5 cuts.
• How can you use exponents rather than writing the factor of 2 over and over again?
MAKING THE MATHEMATICS EXPLICIT
Ask about the pattern in the number of ballots and how it relates to the number of cuts shown in the table, graph, and equation.
Suggested Questions
• What is the growth pattern in the number of cuts and number of ballots situation?
• How does the growth pattern show up in the table? Graph? Equation?
• Is this a linear function? Explain.
• In What If . . . ? Situation A, how many ballots are made after 20 cuts?
• What If . . . ? Situation A asked you to work in reverse to predict the number of cuts needed to make enough ballots for 500 students. Describe your method.
As students are working on the Now What Do You Know? question, have students develop and shares initial draft ideas. The teacher and fellow classmates can identify strengths and areas for potential refinement in the students' draft ideas.
As you finish the mathematical discussions, have students reflect on the Now What Do You Know? question(s).
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Compare Thinking
Agency, Identity, Ownership
Rough Draft Mathematics
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EXTENDED LAUNCH—EXPLORE—SUMMARIZE

Problem Overview

In earlier units, students studied linear functions. This problem introduces a nonlinear function called an exponential function. Students investigate the growth in the number of ballots created by repeatedly cutting a piece of paper in half. By examining the table, they see that, as the number of cuts increases by 1, the number of ballots doubles, or increases by a factor of 2. They also create a graph and write an equation to represent the doubling pattern.

Launch (Getting Started)

Connecting to Prior Knowledge

Remind students that in *Thinking with Mathematical Models*, they looked at two kinds of functions, linear functions and inverse variation.

Suggested Questions

- Give an example of each function. (Answers will vary.)
- What are the variables? (Answers will vary.)
- What is the pattern of change for each function? (Answers will vary.)
- Describe the graphs.
  (Answers will vary for these questions. For linear functions, students might give $d = r \times t$ for the relationship between distance, time, and rate, where d and t are the two variables. For example, $d = 5t$. As t increases, d increases by a constant amount. In the example, d increases by 5 for each one unit increase in t. Students might use these same variables but fix d to represent an inverse variation: $100 = r \times t$. As r increases, t decreases. The graph of a linear function is a straight line, and the graph of inverse variation is curve.)

Presenting the Challenge

Describe the situation with Chen creating ballots for the Student Government Association election.

Suggested Questions

- How can you predict the number of ballots after 8 cuts? (Don't go for closure on this question. Gather a few suggestions, and let them make conjectures.)
- How can you predict how many cuts to make if we need 128 ballots? (Don't go for closure on this question. Gather a few suggestions, and let them make conjectures.)

Later they can compare their predictions to their results from the work in the problem.

Distribute **Learning Aid 1.1: Number of Ballots.** Have students work in groups of 2.

Implementation Note: This problem is an introduction to exponential growth. Students will have time to develop these ideas throughout the unit.

Explore (Digging In)

Providing for Individual Needs

Have paper and scissors available for students to cut the ballots, if needed to see and understand the relationship between the number of cuts and the number of ballots created.

Encourage students to look for the multiplicative pattern in the table.

When students are working on the What If . . . ?, encourage them to look at Monroe's or Dawson's strategy. The teacher can assign partner groups to a strategy. The teacher can differentiate based on the needs of the partner groups. (Agency, Identity, Ownership)

Suggested Questions

- How did you find each of the entries in your table? (Answers will vary. One possible answer: For the first three entries, I found the entries by counting ballots, but after that, I used the doubling pattern.)
- What is the relationship between this number of ballots and the previous number of ballots? (It is twice the previous number.)
- Explain that relationship in terms of the number of cuts. (When the number of cuts increases by 1, the number of ballots doubles.)
- How do you create a graph for this data? (Answers will vary. Help students to see that this isn't linear and that the intervals on the axes need to be constant intervals in order to see the exponential pattern.)
- How could describing the pattern help you to write the equation? (Answers will vary. In the grade 6 unit *Variables and Patterns*, students began writing equations by generalizing an idea in words and then turning these words into an equation. Help students that aren't sure where to begin to think back to generalizing in words before writing the equation.)

Planning for the Summary

What evidence will you use in the summary to clarify and deepen understanding of the Now What Do You Know? question? What will you do if you do not have evidence?

› NOW WHAT DO YOU KNOW?

What are the variables in this situation? How are they related? How is this relationship shown in a table, graph, and equation?

As students work, look for interesting strategies to share during the summary. Students might be using exponential notation rather than listing all the factors of 2 that are needed for each cut. Some will begin to reason using the general relationship between cuts and number of ballots.

Implementation Note: As students are working on the Now What Do You Know? question, have students develop and share initial draft ideas. The teacher and fellow classmates can identity strengths and areas for potential refinement in the students' draft ideas.

Summarize (Orchestrating the Discussion)

Discussing Solutions and Strategies

Have students share their strategies and answers for the problem.

Lead a discussion around exponents. Display a table from one student group. (Agency, Identity, Ownership)

Suggested Questions

- How did you get the number of ballots for 5 cuts? (Most students will say they started with 1 cut and 2 ballots and then multiplied the number of ballots by 2 for each cut until they reached 5 cuts.)

Add a third column to the table, and illustrate each calculation, showing each factor of 2. Stop after showing the calculation for 5 cuts.

Numbers of Cuts	Numbers of Ballots	Calculation
1	2	2
2	4	2 × 2
3	8	2 × 2 × 2
4	16	2 × 2 × 2 × 2
5	32	2 × 2 × 2 × 2 × 2
6	64	
7	128	
8	256	
9	512	
10	1,024	

- How many times is 2 used as a factor to find the number of ballots after 1 cut? After 2 cuts? After 3 cuts? After 4 cuts? After 5 cuts? (once; twice; three times; four times; five times)

- How many factors of 2 will be used to find the number of ballots after 6 cuts? After 10 cuts? After 30 cuts? (six; ten; thirty)

Explain to students that they can use exponents rather than writing the factor of 2 over and over again. Display this equation: $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10}$.

Make sure students see that there are 10 factors of 2 on the left side of the equation and a raised number 10 on the right side. Point to the expression 2^{10}. Explain that the number 2 is the base, the number 10 is the exponent, and the expression 2^{10} is in exponential form.

Explain that 1,024 is the standard form of 2^{10}. Exponential notation was first introduced in the grade 6 unit *Number Connections*.

- Do you have any What If . . . ? questions?
- How do your results compare to the predictions you made earlier? (Answers will vary and depend on the prediction.)
- How is the number of ballots after each cut related to the number of ballots before the cut? (The number of ballots doubles with each successive cut.)
- What does the graph look like? How could you predict this from the table? (The graph is nearly flat at the start, and then it curves and rises steeply. In the table, the numbers are small at first, so doubling is not a very big change. Once the numbers are large, doubling makes a big difference.)

Making the Mathematics Explicit

Ask about the pattern in the number of ballots and how it relates to the number of cuts shown in the table, graph, and equation.

Suggested Questions

- What is the growth pattern in the number of cuts and number of ballots situation? (Each time a cut is made the number of ballots doubles (increases by × 2).)
- How does the growth pattern show up in the table? (As the number of cuts increase by 1, the number of ballots is multiplied by 2.)
- How does the growth pattern show up in the graph? (As the number of cuts increase by 1 (x-axis), the space between the dots for the number of ballots (y-axis) multiplies by 2.)
- How does the growth pattern show up in the equation? (The base of the exponent is a 2, which represents multiplying by 2.)
- Is this a linear function? Explain. (Students should be able to start talking about additive growth for linear functions and multiplicative growth for exponential functions. They may use words like *add a constant* or *the same number* for a linear function

and *multiply by* or *double* for an exponential function. They may point out that the table doesn't increase by a constant rate; the graph is a curve, not a line; and the equation isn't in the form $y = mx + b$.)

- In What If . . . ? Situation A, how many ballots are made after 20 cuts? After 40 cuts? Describe how you found your answers. (220 = 1,048,576. 240 = 1,099,511,627,776. For 20 cuts, I multiplied the number for 10 cuts by itself, which would represent 10 more doublings. For 40 cuts, I multiplied the number for 20 cuts by itself, which would represent 20 more doublings.)
- What If . . . ? Situation A asked you to work in reverse to predict the number of cuts needed to make enough ballots for 500 students. Describe your method. (Students generally find it more difficult to work in reverse, especially because no exact power of 2 is equal to 500. Because 8 cuts gives 256 ballots and 9 cuts gives 512 ballots, 9 cuts are needed to guarantee at least 500 ballots.)

Now What Do Students Know?

Ask students to reflect on the discussion and answer the Now What Do You Know? questions.

See answers to the problem for more information about the strategies and embedded mathematics.

› REFLECTING ON STUDENT LEARNING

Use the following questions to assess student understanding at the end of the lesson.

- What evidence do I have that students understand the Now What Do You Know? question?
  - Where did my students get stuck?
  - What strategies did they use?
  - What breakthroughs did my students have today?
- How will I use this to plan for tomorrow? For the next time I teach this lesson?
- Where will I have the opportunity to reinforce these ideas as I continue through this unit? The next unit?

Answers Embedded in Student Edition Problems

Answers PROBLEM 1.1

Making Ballots: Linear or Nonlinear Relationship?

INITIAL CHALLENGE

After each cut, Chen counts and records the number of ballots in a table.

Number of Ballots for Each Cut

Number of Cuts	Number of Ballots
1	2
2	4
3	
4	
5	

He wants to predict the number of ballots after any number of cuts. So he looks for a growth pattern that describes how the number of ballots changes with each cut.

- Describe a growth pattern Chen might have noticed.

  For each cut, the number of ballots doubles since he cuts them in half: 2, 4, 8, 16, . . .

- How could he use this pattern to predict the number of ballots after 10 cuts?

  He can multiply the previous number of ballots by 2 to get the number of ballots. He can continue making a table for 10 cuts, and the number of ballots will be 1,024.

  Students may discuss that we can continue the mathematical pattern but making 10 cuts with the paper would be difficult.

Number of Ballots for Each Cut

Number of Cuts	1	2	3	4	5	6	7	8	9	10
Number of Ballots	2	4	8	16	32	64	128	256	512	1,024

WHAT IF . . . ?

Situation A. Predicting with a Graph or an Equation

Monroe's Strategy	Dawson's Strategy
I think a graph would be useful for making predictions.	I think an equation would be useful for making predictions.

1. What are the advantages and disadvantages of using a table, graph, or equation to make a prediction?

   Possible answers:

   In a table, it is easy to see the numbers; however, it can be inconvenient that you need to know the previous number to get the next number.

   In a graph, it is easy to see a pattern of growth; however, it might not be easy to draw units on the axes when the numbers get too large.

   In an equation, it is a simple way to represent the relationship; however, it is not easy to see what happens in the relationship.

2. Explain how you would predict the number of ballots made with 20 cuts. Explain how you would predict the number of cuts needed to make 500 ballots.

   Possible answer for 20 cuts:

   You can keep making a table until you get to 20 cuts. Many students will think of this as repeatedly multiplying by 2.

Number of Ballots for Each Cut

Number of Cuts	11	12	13	14	15	16	17	18	19	20
Number of Ballots	2,048	4,096	8,192	16,384	32,768	65,536	131,072	262,144	524,288	1,048,576

   You can also draw a graph and estimate the number of ballots, which might not give the precise number. Or you can use the equation, 2^{20}.

   Possible answer for 500 ballots:

   You can use the table to see that at 9 cuts, there are 512 ballots. On the graph, you can find when the *y*-axis is 500 then move horizontally until you intercept the graph. If you look to see that *x*-axis value, you can see 500 ballots are created at approximately 9 cuts. With the equation, we know that $500 = 2^n$. We can guess and check values of *n* that get us to or near 500. It will take 9 cuts to get 500 ballots and there will be 12 extra ballots.

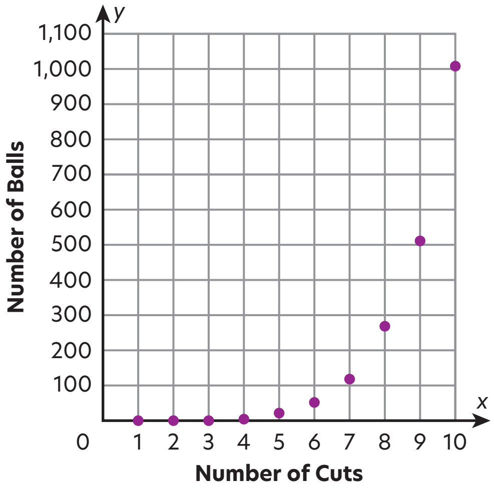
3. How does Chen's growth pattern show up in the table? Monroe's graph? Dawson's equation?

   Table: A multiplying by 2 pattern shows up in the table.

Number of Ballots for Each Cut

Number of Cuts	1	2	3	4	5	6	7	8	9	10
Number of Ballots	2	$2 \cdot 2 = 4$	$2 \cdot (2 \cdot 2) = 8$	$2 \cdot (2 \cdot 2 \cdot 2) = 16$	$2 \cdot (2 \cdot 2 \cdot 2 \cdot 2) = 32$	$2 \cdot (2^5) = 64$	$2 \cdot (2^6) = 128$	$2^8 = 256$	$2^9 = 512$	$2^{10} = 1{,}024$

   Graph: For every increase of 1 cut, the number of ballots doubles. This makes the graph curve steeply. From one point to the next, the vertical distance doubles.



Equation: The repeated multiplying by 2 means that 2 is used as a factor many times, so this can be written as an exponent.

the total number of ballots = $2^{(\text{total number of cuts})}$ or $T = 2^n$

4. How does this relationship compare to a linear function relationship?

   This relationship is not a linear function. In this relationship, as the number of cuts increases by one, the number of ballots doubles, or increases by a factor of 2. In linear functions, as the independent variable increases by 1, the dependent variable increases by a constant amount. The graph of a linear function is a straight line, and the graph of this relationship is a curve, as you can see in Monroe's graph.

> NOW WHAT DO YOU KNOW?

What are the variables in this situation? How are they related? How is this relationship shown in a table, graph, and equation?

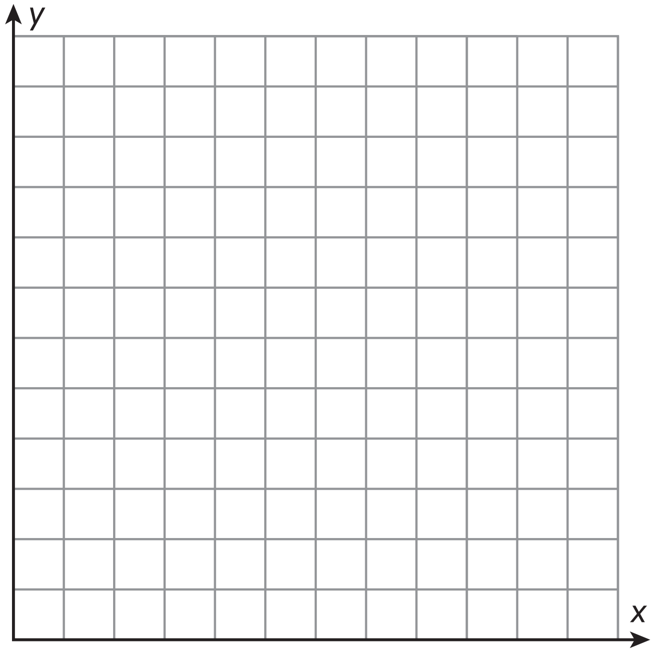
The independent variable is the number of cuts, and the dependent variable is the number of ballots. As the number of cuts increases by 1, the number of ballots doubles. In a table, the numbers get larger than 1,000 after 10 cuts. In a graph, the pattern looks like a curve that goes rapidly up as the number of cuts increases. In an equation, you can see the repeated multiplication of 2 by the number of cuts: $2 \times 2 \times 2 \times 2 \times \ldots \times 2$ or 2^n.

Name ____________________ Date ____________ Class ____________

LEARNING AID 1.1

Number of Ballots

Number of Cuts	Number of Ballots
1	2
2	4
3	
4	
5	
6	
7	
8	
9	
10	



Equation ____________________