UNIT DESCRIPTION

This unit continues the discussion of functions by examining exponential functions. Models of exponential growth and decay are numerous, such as growth or decay of populations—from bacteria, amoebas, radioactive material, and money to mammals (including people). Doubling, tripling, halving, and so on are all intuitive situations for students to help them make sense of exponential functions.

The growth pattern in exponential functions is multiplicative. That is, for each additive change in the independent variable, there is a multiplicative change in the dependent variable. For example, in Problem 1.1, students look at the number of ballots created by repeatedly cutting in half a sheet of paper. As the number of cuts increases by 1, the number of ballots increases by a factor of 2. This factor is called the growth factor.

Investigation 1 continues to look at doubling, tripling, and quadrupling patterns. It ends by contrasting linear and exponential growth factors. It also introduces the y-intercept, or initial value, which it is sometimes called in exponential growth situations. Students often use “starting point” for initial value/y-intercept. Investigation 2 introduces growth factors that are not whole numbers, which then leads to growth rates, usually expressed as percentages. Investigation 3 introduces growth factors that are less than 1 but greater than 0. These are exponential decay situations. The investigation ends by looking at the effects of growth factors and y-intercepts on graphs of exponential functions. In Investigation 4, patterns with exponents are explored and connected to scientific notation and exponential functions.

Exponential growth patterns can grow rapidly. Students may encounter answers on their calculators expressed in scientific notation. Therefore, scientific notation is introduced in Investigation 1 and used throughout the unit with the exception of Investigation 3.

As with all of the *Connected Mathematics*® 4 units, one Mathematical Reflection guides the development of the understanding of the mathematical ideas in the unit.

Mathematical Reflection

In this unit, we are exploring an important type of nonlinear function, exponential functions. At the end of this investigation, ask yourself:

What do you know about exponential functions and exponential expressions? How do exponential functions compare to linear functions?

SUMMARY OF INVESTIGATIONS

Investigation 1: Exponential Growth Patterns

In Investigation 1, students explore situations that involve repeated doubling, tripling, and quadrupling. Students are introduced to one of the essential features of many exponential patterns: rapid growth. They make and study tables and graphs for exponential situations, describe the patterns they see, and write equations for them, looking for a general form of an exponential equation. They also compare and contrast linear and exponential patterns of growth.

In Problem 1.1, students create ballots by repeatedly cutting a piece of paper to create the ballots. They identify the variables and begin to make conjectures about the relationship between the two variables as represented in tables and graphs. Problem 1.2 is based on an old fairy tale. The king rewards a peasant for saving his daughter. The peasant requests a reward based on a chessboard: placing 1 ruba on the first square of a chessboard, 2 rubas on the second square, 4 rubas on the third square, and so on. The students discover this is more money than in the treasury of the country. The problem investigates other reward plans and compares them. Three of the reward plans are exponential growth and one is linear growth. Comparing multiplicative growth patterns (exponential functions) with additive growth patterns (linear functions) reinforces understanding of each function. Problem 1.3 is based on a real-life experience, the growth of water hyacinths in Lake Victoria. It introduces an exponential relationship with y-intercepts greater than 1.

The standard form of an exponential equation is $y = a(b^x)$. When $x = 0$, the equation becomes $y = a$ since $b^0 = 1$. Thus a, the coefficient of the exponential term, generally indicates the initial value of the exponentially growing quantity. The growth factor is b. As the value of x increases by 1, the value of y increases by a factor of b. By contrast, a linear function is represented by the equation $y = mx + b$, where m is the constant rate of change (or slope) and b is the y-intercept. In the What If . . . ?, the growth of snakes is represented by a graph from which students find the y-intercept and growth factor. In Problem 1.4, students use an experiment of growing mold. They continue to write an equation that models the data and how the equation represents the initial population and growth factor. Students explore situations given in context, tables, graphs, and equations and determine whether the situation represents an exponential function by identifying the growth factor. They use the growth factor and y-intercept to write an equation. Since exponential growth represents very large numbers quickly, scientific notation is introduced in this investigation to make predictions, and its use continues throughout the unit.

Investigation 2: Growth Factors and Growth Rates

In Investigation 2, students study fractional growth factors greater than 1. They relate these growth factors to growth rates. Problem 2.1 provides data for the growth of rabbits in Australia. Students examine the data and find that the growth factor in not a whole number. They use the growth factor to make predictions about the population. Some growth patterns, such as investments, are often expressed as percents. Problem 2.2 provides the context of college savings in bonds that grow at a certain percent. This means that to find the growth rate, the percent increase is calculated and added to the initial, or previous, value. The growth factor is determined by dividing the next value by the previous value. For example, if $100 is invested at a 6% annual interest rate, the value of the account at the start of the first year is 100 + 100(0.06) or 100(1.06). The *growth factor* in this case is 1.06 while the *growth rate* is 6%, or 0.06. This type of change is called *compound growth*. Students also explore how the growth rate and the initial value affect the growth pattern. The context of examining the growth of different species of antelope provides an opportunity to contrast linear and exponential functions and to explore growth factors and growth rates. In Problem 2.3, they continue to explore growth factors and growth rates in the context of savings accounts. They end the problem by looking at the growth of algae in a lake and again look at the power and complexity of exponential growth. Scientific notation continues to be used throughout, as do exponents, which are needed to write equations for exponential functions.

Investigation 3: Exponential Decay

Investigation 3 introduces students to exponential decay—patterns of change that exhibit successive, nonconstant decreases rather than increases. These decreasing relationships are generated by repeated multiplication by factors between 0 and 1, called decay factors. Strategies for finding decay factors, for finding the initial population, and for representing decay patterns are similar to those used for exponential growth patterns. Exponential decay patterns are also represented by the equation $y = a(b^x)$; a is the y-intercept, and b is the growth factor, which is greater than 0 and less than 1.

In Problem 3.1, creating ballots is revisited, but this time students look at what is happening to the area of each ballot. The area decreases by a factor of $\frac{1}{2}$. They look at the relationship between the variables in a table, graph, and equation. They notice that the pattern of growth as represented in the table is growing by a factor less than 1 and notice how this affects the graph and equation. In Problem 3.2, students continue to look at exponential decay in the context of a growth of fleas. With a certain medication, the fleas decrease exponentially, which is represented in a table and graph. This introduces the use of percents to represent decay. In Problem 3.3, students use the experiment of cooling water to study exponential decay. In Problem 3.4, students use graphing calculators and/or online graphing tools to study the effects of the values of *a* and *b* on the graph of $y = a(b^x)$ to summarize the exploration of a type of nonlinear function, exponential functions.

Investigation 4: Patterns with Exponents

This investigation develops properties for operating with exponents. In Problem 4.1, students examine patterns in a powers table for b^x for any *b* and $x = 1, 2, 3, 4, 5, 6, 7, 8, 9$. They look for relationships among numbers written in exponential form. The patterns they observe lead to the properties for operating on numerical expressions with exponents. In Problem 4.2, students continue to look for and refine patterns in the powers table. They provide arguments for why their patterns work and conclude with several important properties of exponents: $a^m \times b^m = (ab)^m$, $\frac{a^m}{a^n} = a^{m-n}$, $a^m \times a^n = a^{m+n}$, $a^0 = 1$, $(a^m)^n = a^{mn}$, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ and $a^{-m} = \frac{1}{a^m}$. They use these properties to revisit some of the previous contexts in the unit.

In Problem 4.3, the properties for exponents are used to write and interpret equivalent expressions, including some expressed in scientific notation. The context is the amount of water used for various activities in the United States, such as water needs for power plants, livestock, and irrigation. This problem ends with the context of populations of hogs and pigs in the United States, which involves exponential functions with scientific notation. In Problem 4.4, rational exponents are introduced by looking at graphs of populations of amoebas and asking questions about the population at $\frac{1}{2}$ a month or $\frac{5}{3}$ of a month or when $x = \frac{1}{2}, \frac{5}{3}$, and so on. The *n*th roots are used to interpret and evaluate expressions with rational exponents. From these explorations, students observe that the properties for integral exponents apply to rational exponents.