MATHEMATICS OVERVIEW

Major Focus: variables, expressions, equations, algebraic reasoning

Key Ideas: variable, independent and dependent, relationship, expression, equation, evaluate, substitute, solve/solution

Strategic Curriculum Connection: proportional reasoning

Algebra has its roots and applications in a variety of human activities from commerce to the study of numbers, from the expression of patterns of change to the elaboration of other branches of mathematics. The centrality and importance of algebra in mathematics reflects its power to compactly and efficiently express mathematical structures and patterns within and across strands. In the Connected Mathematics Project® (CMP®), the development of algebra evolves as a language to express important mathematical concepts in a more general way. Through the efficient and compressed symbol systems of algebra, deep yet simple mathematical structures and patterns can be represented.

> It is essential for students to learn algebra both as a style of mathematical thinking, involving the formalization of patterns, functions, and generalizations, and as a set of competencies involving the representations of quantitative relationships. (Silver 1997)

Connected Mathematics® is a problem-centered curriculum in which quantities or variables naturally arise in the context of a problem. In this curriculum, it makes sense to think about how variables are related, how they can be represented, and the information we get from the way they are represented. The ten *Connected Mathematics®* algebra and functions units challenge students, through a carefully sequenced set of problems, to represent algebraic relationships in words, tables, graphs, and symbols. Time is spent exploring interesting mathematical situations, comparing representations, and reflecting on solution methods. This allows students to build robust understandings of what it means to write and reason with symbolic expressions.

Patterns of change or functions naturally occur as students identify the variables in a situation and how they are related. Equations are first understood as symbolic rules that relate independent and dependent variables. One-variable equations are, therefore, specific instances, or snapshots, of function relationships. This curriculum's approach intertwines reasoning about algebra and functions into one coherent algebra and functions strand across three years.

A natural outcome of combining functions and algebra through problem-solving situations is that one-variable equations, linear and nonlinear, can be solved with tabular and graphical methods and numerical reasoning, as well as symbolically. Manipulation of symbolic expressions into equivalent expressions is viewed as a tool to reveal new insights about a relationship that the equivalent symbolic expressions represent in the problem situation. Examining and refining strategies leads to new strategies and a deeper understanding of expressions and equations. By deploying a functions approach to algebraic reasoning, students see both algebra and functions as a coherent whole.

The approach in *Connected Mathematics®* to algebra has been extensively researched, including a longitudinal study done by Cai and coauthors (2011) that followed eighth-grade students who did and did not follow *Connected Mathematics®* from their eighth grade through their senior year of high school. Students who followed *Connected Mathematics®* did better on skills and problem solving and maintained a more positive and flexible approach to mathematics through high school.

In the *Variables and Patterns* unit, using the theme of planning a bike trip, students write and interpret graphs and tables that represent the relationships between two variables. In later problems, they graph equations that capture relationships between the two variables, and they also write equations of the form $y = mx + b$. In this unit, the contexts are of the form $y = ax$ or $y = x + b$. The equations $y = ax$, or $d = 50t$, that occur in this unit are also beginning to explore proportional reasoning. This is not the focus in this unit. However, by making tables and graphs of $y = ax$, one aspect of proportional reasoning is emerging: how a proportional relationship appears in table and graph representations. This will be addressed using the appropriate vocabulary in the *Comparing Quantities* unit.

Link to the Future

Because this unit is the first in grade 6, all subsequent units in grades 6, 7, and 8 will use variables in different contexts: representing strategies and patterns in algebraic language. For example:

- Representing multiples of 5 as $5M$, where M is a whole number (*Number Connections: Expressing Factors and Multiples Algebraically* unit)
- Exploring tables and graphs of proportional reasoning (*Comparing Quantities* unit)
- Area A of a triangle as $A = \frac{1}{2}bh$ with base b and height h (*Covering and Surrounding* unit)
- Representing multiplication of rational numbers and solving equations of the form $ax = b$ or $x + a = b$ (*Bits of Rational* and *Points of Rational* units)

At grade 6, students solve only simple additive and multiplicative equations. They also use algebraic language to represent patterns in number and geometry. In grade 7, grade 8, and high school, students will study more complex predictable relationships, such as linear, quadratic, exponential, inverse, and other predictable relationships.

In roughly this order, in *Variables and Patterns*, students:

- represent and **analyze relationships** between variables in words, symbolic statements (equations), graphs, and tables;
- **write, read, and evaluate** the variable expressions that form the symbolic representations of relationships and use the representations to make predictions and solve problems; and
- understand the meaning of the equals sign in an equation, **solve equations**.

Represent and Analyze Relationships, Tables, and Graphs

We start with a messy and realistic situation, jumping jacks, which cannot be easily represented with a simple equation. The goal is to introduce the idea of *variables*, in this case time, *t*, and number of jumping jacks, *n*. As time changes/varies, the cumulative number of jumping jacks increases/varies. There is a *relationship,* but it is not a straightforward pattern, such as two jumping jacks for every 1 second. While both are variables, time is beyond our control; it changes whether jumping jacks happen or not. Time is the *independent variable*. The cumulative number of jumping jacks depends on how many seconds have passed; this is the *dependent variable*. The relationship can be *represented* in words, in a table, and on a graph.

Written Description

The description gives an overall idea of the relationship between the variables. However, we do not know specific information.

Ana's Group

Ana had a pace that seemed to change every 10 to 20 seconds. In the beginning, the total number of jumping jacks increased a lot. Then she slowed down. Then she sped up and finally she slowed down toward the end.

Table

The table gives Ana's data that can be used to see the relationship between the variables. However, there is no easy pattern or formula.

Table 6

Time (seconds)	0	10	20	30	40	50	60	70	80	90	100	110	120
Total Number of Jumping Jacks	0	12	24	30	35	48	59	62	65	68	80	93	107

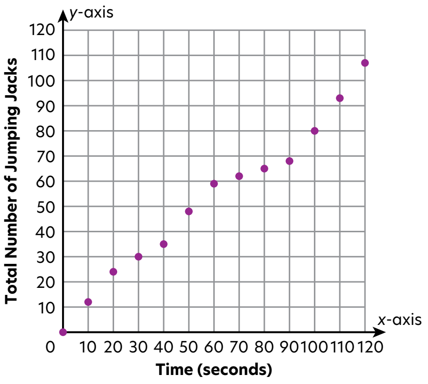
Graph

The graph visually shows the trend in Ana's jumping in the data. However, interpreting the trends

Equation

It is not possible to write a simple equation.

Link to the Future: The pattern in the graph seems to have three different parts: a moderate start, a slow middle, and a sprint at the requires some experience, like being able to determine when the relationship between the variables was increasing the most or least or staying the same. Also, some points may be estimates, the points that do not fall exactly on a grid line.



end. Students will learn in the *Function Junction* unit (grade 8) and high school algebra to write piecewise equations for this kind of situation.

Rationale for Curriculum Decision

Why do we start with a messy situation? We don't want to imply to students that math is only useful in simplified, unrealistic situations. Instead, we want students to think of the big picture, relationships, before we think of specific relationships, like $y = x + a$ and $y = bx$. Students need a road map for the journey through *all of algebra*, one that can zoom out to see the connections and zoom in to see the details.

Read, Write, and Evaluate Variable Expressions

The simplest relationships to explore in the beginning are linear. A *linear relationship*, where the dependent variable is increasing at a constant rate while the independent variable increases, can be captured in a table and graph but also in an equation. For example, distance traveled, D, increases at a constant rate as time, t, increases, if the speed is constant. Students can analyze the entries in a table and look for a numerical pattern. They *write an expression* for the numerical pattern they see, using *variables*. They can *evaluate this expression* for other values of t.

Written Description

Someone travels 50 miles for every hour, or 50 miles per hour, so distance, D, increases in a regular way when time increases.

Table

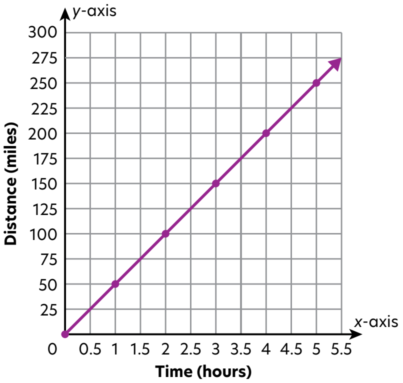
For every hour traveled, the total distance increases by 50 miles.

Time (hours) (t)	Distance for Speed of 50 mph (d_1)
0	0
1	50
2	100
3	150
4	200
5	250
⋮	⋮

Graph

For every hour traveled, the total distance increases by 50 miles.

This consistent pattern of increase creates a graph where the points line up.



Equation

For every hour traveled, the total distance increases by 50 miles.

The pattern can be represented by an equation.

Distance traveled is 50 times the number of hours: $D = 50t$.

The expression $50t$ always gives us a value for D if we *substitute* different values for t. We can *evaluate the expression* $50t$ for values of t not given in the table.

Students also learn *to read and make sense of symbolic statements*. For example, what question are we answering when we solve $300 = 50t$?

What question are we asking with the statement $D = 50(2.5)$? They learn *to interpret symbolic statements*. For example, what situation might be described by $N = 2t$? (Perhaps this describes the total number of jumping jacks if the jumper can do 2 per second. Or meters traveled at 2 meters per second. Or the cost of iced teas at $2 each.)

Strategic Curriculum Connection

In discussing the difference between the graphs for the jumping jacks and these graphs, students might talk about the constant/regular increase. This is *not* the major focus of the unit, but it foreshadows proportional reasoning and linear functions. Language like "For every 1 hour, the distance increases by 50 miles" is foreshadowing ratio and linear functions. In the jumping jack experiment, there is no constant rate. If students use this language, the teacher might draw attention to it. It is not something to hold students accountable for or to assess, just to encourage.

Solving Equations

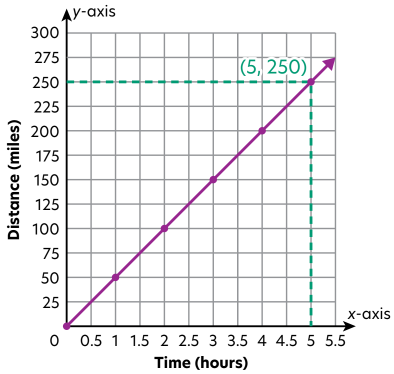
Many people have encountered *solving equations* as a set of separate skills to be mastered. There may have been no meaning within the contextless equation and little meaning in the procedures involved. In *Variables and Patterns,* each equation is seen as a specific instance of a relationship. The tools students use to solve equations are the same tools they developed to analyze the relationship: tables and graphs.

Suppose the question is "How long will it take to drive 250 miles, if the car is traveling at 50 miles per hour?" The relationship is $D = 50t$. The specific equation we are asked to solve is $250 = 50t$.

Since we can represent the relationship between time and distance in a table, we can also use the table to find the solution to *the related equation.*

Time (hours) (t)	**Distance for Speed of 50 mph (d_1)**
0	0
1	50
2	100
3	150
4	200
5	250
⋮	⋮

Since we can represent the relationship between time and distance on a graph, we can also use the graph to *solve the related equation.*



The key understanding is that, in this situation, *D* and *t* are always related by the equation $D = 50t$. When we are given a value for one variable, we can use the equation—or the table or the graph—to find the value for the other variable. We can do this by *substituting values for t* into the equation $250 = 50t$ until you find the value of *t* that *makes the equation true*, or you can scan a table or graph looking for the *related pair* (*t*, 250). Of course, the same table and graph can be used to solve an infinite number of other equations, so long as they are specific instances of the relationship $D = 50t$: $220 = 50t$, $1{,}000 = 50t$, and so on.

Rationale for Curriculum Decision

The virtue of embedding solving equations into the study of relationships is that these tools, tables and graphs, can be used to solve *any* equation involving real numbers, with real solutions. Traditionally students were taught only symbolic methods. Each new type of equation required a different strategy. Perhaps no one ever shared with students that there are many equations that *cannot* be solved by a symbolic strategy. Graphical and tabular methods work for *any* equation with a real solution.

Link to the Future: In the unit *Bits of Rational*, later in grade 6, students explore symbolic ways to solve simple linear equations: for example, using fact families or undoing an operation. In the grade 7 units *Comparing and Scaling* and *Moving Straight Ahead*, students solve linear equations of more complexity, for example, $2x = 3x - 5$, $-4.7x - 6.3 = 3(2x + 0.8)$. In grade 8, students learn about quadratic equations: for example, $x^2 = 2x + 3$. While symbolic methods are developed for solving some nonlinear equations, tables and graphs can still be used. As students continue in their algebraic careers, they will encounter equations that cannot be solved with symbolic methods; for example, $x^3 = 2x + 3$. Tables and graphs work for all equations with real solutions. In undergraduate mathematics, there are courses devoted to developing estimation techniques to solve equations when symbolic algorithms do not exist.

References

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